Lecture 11. **Direct estimation method of quality of regulation. Root estimation method of quality of regulation**

11.1 Direct estimation method of quality of regulation

Let us consider a system described by a set of linear differential equations with constant coefficients, and let the unit step function be applied on the system input:

$$g(t) = 1(t) = \begin{cases} 1 \;\; by \;\; t > 0 \\ 0 \;\; by \;\; t \le 0 \end{cases}.$$

Initial conditions are zero. The system response in this case will be the transient function *h(t).*

Transients that occur in systems under action of step disturbances are usually classified into 3 groups: a) monotonous, b) aperiodic and c) oscillating (examples are given in fig. 4.4).

We will consider the general case of a transient process (fig. 4.5) and a transient process with deviations (fig. 4.6)

There is a group of direct quality indices which deal with the transient curve. Namely: regulation time, overshoot, number of oscillations, damping factor, steady displacement (that determines steady-state accuracy of the system).

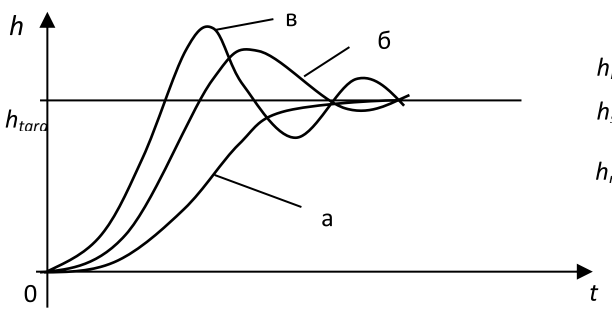


Fig. 4.4. Different transient types

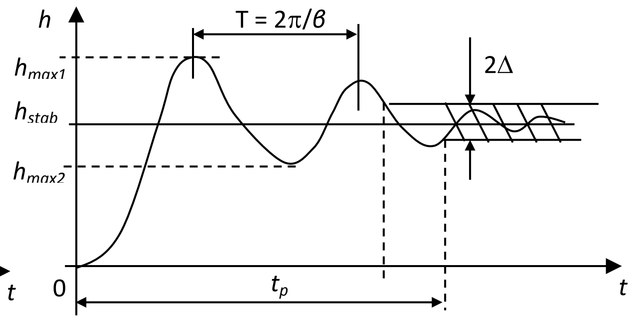


Fig. 4.5. General view of a transient oscillating process

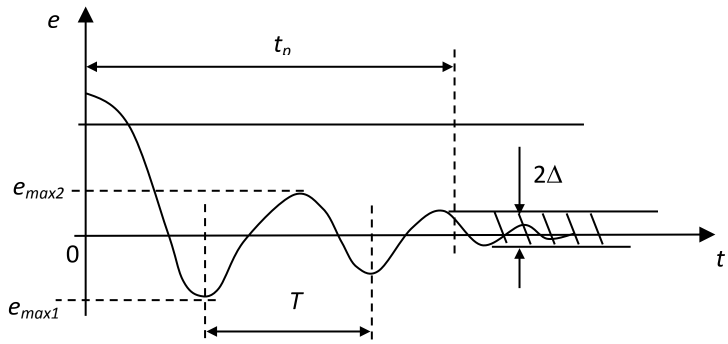


Fig. 4.6. Deviation transient of the oscillating process

Consider all of them in more details:

1) *Regulation time* t_P, characterizes operation speed of a system; it is the minimal time that is required to obtain output coordinate sufficiently close to the target value with the needed precision:

$$|h(t) - h_{stab}| \le \Delta, \qquad (4.3)$$

where Δ is predefined and means allowable error (in practice it is usually set equal to 5% of steady-state value h_{stab}).

Condition (4.3) is required to compute regulation time by using the formula

$$t_p = k\,dt,$$

where "*dt*" is the sampling interval of a continuous signal, k is the number of intervals.

2) *Overshoot* σ is the maximal allowable relative deflection of transient characteristic from the steady-state value of the output coordinate, measured in percents or relative units:

$$\sigma = \frac{h_{\max 1} - h_{stab}}{h_{stab}} * 100\% \qquad (4.4)$$

If a characteristic is the deflection one, we will have:

$$\sigma = \frac{|e(t)|_{MAX1}}{|e(0)|} * 100\%\ , \qquad (4.4a)$$

where $e(0) = \Delta$.

It is common to use 10-30% as the allowable overshoot value, but larger values can be met, for instance 70%. Nevertheless, there exist object that do not tolerate overshoot at all, that is $\sigma = 0$.

3) *Number of oscillations* $h(t)$ that transient make during regulation time t_P.

Often in ACS number of oscillations is allowed be $m = 1 \div 2$, less frequently $m = 3 \div 4$. Again, some systems do not permit oscillations at all.

4) *Damping factor* μ is equal to the ratio between absolute values of two adjacent overshoots, that is:

$$\mu = \frac{|h_{MAX\,1} - h_{stab}|}{|h_{MAX\,2} - h_{stab}|} = \frac{|e_{MAX\,1} - e(0)|}{|e_{MAX\,2} - e(0)|} \qquad (4.5)$$

where $e(0) = \Delta$.

5) Steady *displacement* is computed in the following manner:

$$\delta_{stat} = \frac{\Delta}{h_{t\arg}} * 100\% = \frac{\Delta}{h_{stab} - \Delta} * 100\% \quad , \qquad (4.6)$$

where h_{targ} is the desired value. It determines steady-state precision of a system.

All aforementioned terms are not the only possible and their list can be easily extended to include some other, more specific ones; the exact indices to use are determined by the system under consideration and the task to fulfill.

11.2 Root method of quality estimation

As it was shown previously, roots of characteristic equation determine system transient behavior. Hence, it is possible to produce constraints on stability margin and operation speed of a system by dealing with these roots, not with the transient itself.

Let a characteristic equation of a system be of the following form:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = 0 \quad . \tag{*}$$

Geometric mean of the root Ω_0 by definition is

$$\Omega_0 \overset{\Delta}{=} \sqrt[n]{|s_1 * s_2 * \ldots * s_n|} = \sqrt[n]{\frac{a_n}{a_0}} \quad . \tag{4.7}$$

Speed of the transient flow can be measured by this value; hence it can be used to estimate the system speed of operation. According to (4.7) to increase the value Ω_0 we need to increase the value of free term a_n. In static systems the free term is equal to $a_n = (1 + K_0)$, in astatic is equal $a_n = K_0$, where K_0 is the overall gain factor.

Consequently, an increase of the speed of transient process can be realized by increasing of the coefficient, which is equal to $K_0 = \prod_{i=1}^{n} k_i$, where k_i are link gain factors.

At the same time we need to keep in mind stability of the system, i.e. if the condition $K_0 < K_{cr}$ is satisfied. Here K_{cr} is a boundary increasing factor *(critical).*

There exists one useful notion that allows estimating speed of operation, namely *degree of stability* η*.*

It is defined as the absolute value of real part of the root closest to the imaginary axis: $\eta = |\mathrm{Re}\, s_{\min}|$ (fig. 4.7).

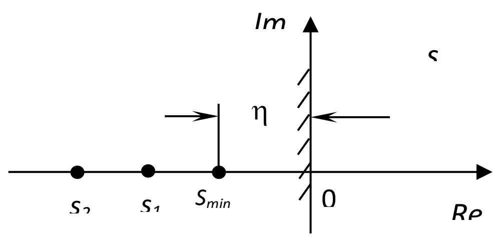


Fig. 4.7. Illustration of degree of stability concept

This value η determines decay speed of transient process, since it ends up when the term defined by the abovementioned root (closest to the imaginary axis) have decayed.

From this we can approximate dependence between the degree of stability and the transient time:

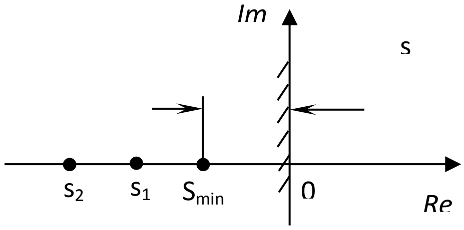
$$t_P \approx \frac{1}{\eta} * \ln \frac{1}{\Delta},$$

where $\Delta = (0{,}01 \div 0{,}05)h_{stab}$ and is usually predefined.

Several cases are possible:

<u>*The first case:*</u> *The closest root is real*, that is $s = -\alpha$.

In this case two roots have to be considered, s_{min} and s_1; (fig. 4.8a), then $T = \frac{1}{s_{\min}}$, here "T" is the time constant, and $\tau = \frac{1}{|s_1|}$, where τ is the time required to reach settled initial conditions. In this case transient process time is $t_P \approx (3 \div 4)T$ (fig. 4.8b).



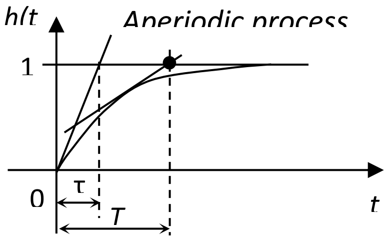
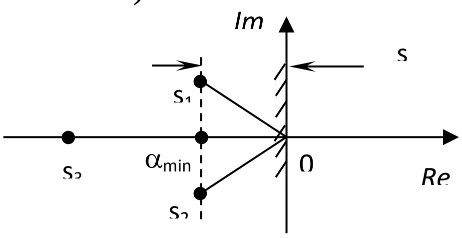


Fig. 4.8a. A real root

Fig. 4.8b. The transient process

<u>*The second case:*</u> *The closest roots are conjugate*, that is, $s_{1,2} = -\alpha \pm j\beta$, impacts on transient of both these roots are equal. The transient time is $t_P = \frac{1}{|\alpha_{\min}|}$ (fig. 4.9a. and 4.9b).



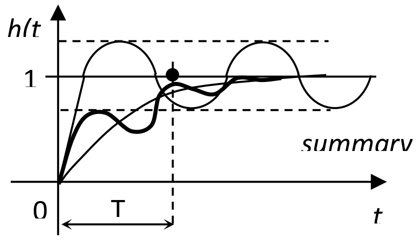


Fig. 4.9a. Conjugate roots

Fig. 4.9b. The corresponding transient

Now it is the best time to explore the degree of stability notion more deeply. It was introduced by russian scientists *Y. Z. Tzipkin* (*Я.З. Цыпкин*) and *P. W. Bromberg* (*P.B. Бромберг*). Its important feature is there is no need to calculate characteristic roots in order to find its value. Another approach is applicable, when we introduce new variable $z = s + \eta$ and substitute into (*) $z = s - \eta$ thus obtaining mixed equation:

$$a_0(z-\eta)^n + a_1(z-\eta)^{n-1} + \ldots + a_{n-1}(z-\eta) + a_n = 0 .$$

After opening the brackets and grouping similar terms together we obtain the next equation (with left shift by the value η):

$$a_0 z^n + A_1 z^{n-1} + \ldots + A_{n-1} z + A_n = 0 \ , \qquad (4.8)$$

where $A_n = a_n - a_{n-1}\eta + a_{n-2}\eta^2 - a_{n-3}\eta^3 + \ldots$ (here signs are alternating in series).

Next, to compute the value of degree of stability it is sufficient to apply any stability criterion on equation (4.8) and determine at what values of η the stability threshold occurs. Just to remind: aperiodic threshold occurs when the free term (A_n in this case) is equal to 0; oscillating threshold is signified by zero next to last Hurwitz determinant.

<u>The third case</u>: *The closest roots are conjugate*, that are $s_{1,2} = -\alpha \pm j\beta$; but the second term's impact is greater than the first's one.

Here $\gamma = \beta / \alpha$ is the system oscillation (variability) and α, β are real and imaginary parts of the root (respectively) in fig. 4.10a.

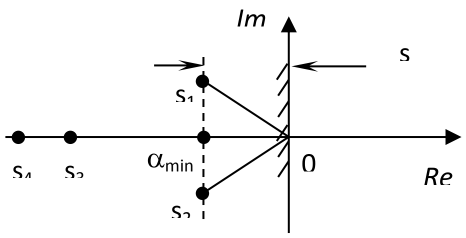


Fig. 4.10a. Conjugate roots

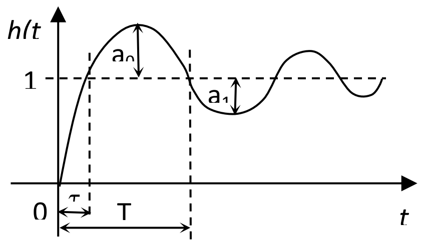


Fig. 4.10b. Corresponding transient

Oscillation is closely tied to another root characteristic of stability margin, namely to decay. Decay during period is equal to $T = \dfrac{2\pi}{\beta}$, where β is an imaginary part of the closest root.

Damping factor is defined as

$$\mu = \frac{a_0 - a_1}{a_0} = 1 - \frac{a_1}{a_0}$$

and is measured in percents % (fig. 4.10b).

As a rule, in ACS decay during period is allowed to be not less that 90-98%.

Defining particular oscillation applies constraint on roots position in complex plane: the region which roots can occupy is restricted by two rays that make with real axis angle

$$\varphi = arctg \frac{\beta}{\alpha} = arctg \ \gamma \ ,$$

where α and β are real and imaginary parts of the closed to the axis root of characteristic equation.

Oscillation of a system can also be computed without calculating characteristic roots, in the way similar to the one described in the second case.

Transient is a subject to applied constraints:

if $\varphi_{\min} \leq \varphi_{доп}$, then transient function *h(t)* is the monotonous one and complex roots can be neglected (fig. 4.11a. and 4.11b).

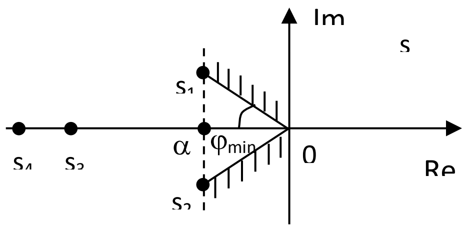


Fig. 4.11a. Allowable region for roots

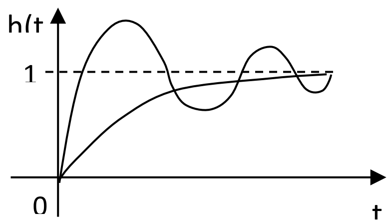


Fig. 4.11b. The transients

In general, usage of characteristic equation roots for quality estimation is not complete, since the transient behavior follows not only from left-hand side, but from right-hand side of the differential equation as well. By definition, *left-hand* side (transfer function denominator) roots are *poles*; *right-hand* side (transfer function numerator) roots are noughts.

If we define fields of poles and noughts placement we can estimate transient behavior more complete, for example, zeroes accelerate transient. Noughts effect the transient increasing.

Note: system lag can be decreased in steady state, if we situate noughts around poles of the transfer function, starting from the point closest to the imaginary axis.